A NEW PROOF OF LUCAS' THEOREM

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Abstract. We give a new proof of Lucas' Theorem in elementary number theory.

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1. Introduction

One of the most useful results in elementary number theory is the following result of E. Lucas

Theorem 1.1 ([1],E. Lucas (1878))**.** *Let p be a prime and m and n be two integers considered in the following way,*

$$m = a_k p^k + a_{k-1}p^{k-1} + \ldots + a_1 p + a_0,$$
$$n = b_l p^l + b_{l-1}p^{l-1} + \ldots + b_1 p + b_0,$$

where all a_i and b_j are non-negative integers less than p. Then,

$$\binom{m}{n} = \binom{a_k p^k + a_{k-1}p^{k-1} + \ldots + a_1 p + a_0}{b_l p^l + b_{l-1}p^{l-1} + \ldots + b_1 p + b_0} \equiv \prod_{i=0}^{\max(k,l)} \binom{a_i}{b_i} \pmod{p}.$$

Notice that the theorem is true if $a_i \geq b_i$ for $i = 0, 1, 2, \ldots, \max(k, l)$

There has been many different proofs of this result in the years that followed its first publication. We present here an alternate approach using elementary number theoretic techniques.

2. Proof of Theorem 1.1

First of all, we state and prove a few lemmas

Lemma 2.1. *If*

$$a_0 + a_1 X + a_2 X^2 + \ldots + a_n X^n \equiv b_0 + b_1 X + b_2 X^2 + \ldots + b_n X^n \pmod{p}$$

then

$$a_i \equiv b_i \pmod{p} \ \forall\, i \in [[0, n]].$$

Proof. Indeed, if $a_0 + a_1 X + a_2 X^2 + \cdots + a_n X^n \equiv b_0 + b_1 X + b_2 X^2 + \cdots + b_n X^n \pmod{p}$, then there exits a polynomial $k(X) = k_0 + k_1 X + k_2 X^2 + \cdots + k_n X^n$ at most of degree n such that $a_0 + a_1 X + a_2 X^2 + \cdots + a_n X^n = b_0 + b_1 X + b_2 X^2 + \cdots + b_n X^n + p(k_0 + k_1 X + k_2 X^2 + \cdots + k_n X^n)$. This gives $a_0 + a_1 X + a_2 X^2 + \cdots + a_n X^n = b_0 + pk_0 + (b_1 + pk_1)X + (b_2 + pk_2)X^2 + \cdots + (b_n + pk_n)X^n$. Hence we get $a_0 = b_0 + pk_0$, $a_1 = b_1 + pk_1$, $a_2 = b_2 + pk_2$,..., $a_n = b_n + pk_n$. Or equivalently $a_0 \equiv b_0 \pmod{p}$, $a_1 = b_1 \pmod{p}$, $a_2 = b_2 \pmod{p}$,$\cdots$, $a_n = b_n \pmod{p}$.

The reciprocal implication is trivial. □

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Lemma 2.2. *If the base p expansion of a positive integer n is,*

$$n = a_0 + a_1 p + a_2 p^2 + \cdots + a_l p^l$$

then we have

$$n! = q a_0!(a_1 p)!(a_2 p^2)! \cdots (a_l p^l)!$$

with q a natural number.

Proof. Since the factorial of a natural number is a natural number, there exists a rational number q such that

$$q = \frac{n!}{a_0!(a_1 p)!(a_2 p^2)! \ldots (a_l p^l)!}.$$

Let S be the set,

$$S = \{x_1, x_2, \ldots, x_l\}.$$

We consider lists of elements of S where x_0 is repeated a_0 times, x_1 is repeated $a_1 p$ times,..., x_l is repeated $a_l p^l$ times such that, $0 \leq a_i \leq p-1$ with $i \in [[0, l]]$. In such a list, there are $l+1$ unlike groups of identical elements. For instance the selection $\left(\underbrace{x_0, x_0, \ldots, x_0}_{a_0}, \underbrace{x_1, x_1, \ldots, x_1}_{a_1 p}, \ldots\ldots, \underbrace{x_l, x_l, \ldots, x_l}_{a_l p^l}\right)$ is such a list of n elements which contains $l+1$ unlike groups of identical elements.

The number of these lists is given by $\frac{n!}{a_0!(a_1 p)!(a_2 p^2)!\ldots(a_l p^l)!}$. It proves that the rational number q is a natural number. And, since the factorial of a natural number is non-zero (even if this number is 0 because $0! = 1$), we deduce that q is a non-zero natural number. □

Theorem 2.3. *Let $n = ap + b = a_0 + a_1 p + a_2 p^2 + \ldots + a_l p^l$ such that $0 \leq b \leq p-1$ and $0 \leq a_i \leq p-1$ with $i \in [[0, l]]$. Then $q \equiv 1 \pmod{p}$.*

Before we prove Theorem 2.3 we shall state and prove the following non-trivial lemmas.

Lemma 2.4. *The integers q and p are relatively prime.*

Proof. If $0 < q < p$, since p is prime, q and p are relatively prime.

If $q \geq p$, let us assume that p and q are not relatively prime. It would imply that there exist an integer $x > 0$ and a non-zero natural number q' such that $q = q' p^x$ with $\gcd(q', p) = 1$. Since $n! = q a_0!(a_1 p)! \ldots (a_{l-1} p^{l-1})!(a_l p^l)!$ we get $n! = q' p^x a_0!(a_1 p)! \ldots (a_l p^l)!$. It follows that $n!$ would contain a factor $a_{l+x} p^{l+x}$ such that $q' = a_{l+x} q''$ with $a_{l+x} \in [[1, p-1]]$. But, $a_{l+x} p^{l+x} > n$.

Indeed we know that $1 + p + \ldots + p^l = \frac{p^{l+1}-1}{p-1}$. So $p^{l+1} = 1 + (p-1)(1 + p + \ldots + p^l)$. Then $p^{l+1} > (p-1) + (p-1)p + \ldots + (p-1)p^l$. Since $0 \leq a_i \leq p-1$ with $i \in [[0, l]]$, we have $0 \leq a_i p^i \leq (p-1)p^i$ with $i \in [[0, l]]$. Therefore, $p^{l+1} > a_0 + a_1 p + \ldots + a_l p^l$ so $p^{l+1} > n \Rightarrow a_{l+x} p^{l+x} > n$.

Since $n!$ doesn't include terms like $a_{l+x} p^{l+x} > n$ with $a_{l+x} \in [[1, p-1]]$, we obtain a contradiction. It means that the assumption $q = q' p^x$ with $x \in \mathbb{N}^\star$ and $gcd(q', p) = 1$ is not correct. So, q is not divisible by a power of p. It results that q and p are relatively prime. □

We know that $n! = (ap + b)! = q a_0!(a_1 p)! \ldots (a_l p^l)!$ with $a = \lfloor \frac{n}{p} \rfloor$, $0 \leq a_i \leq p-1$ with $i = 0, 1, 2, \ldots, p-1$ and $b = a_0$. Let $q_{a,l,1,i}$ with $0 \leq i \leq a_1 \leq a$ be the natural number

$$q_{a,l,1,i} = \frac{(ap + b - ip)!}{a_0!((a_1 - i)p)!(a_2 p^2)! \ldots (a_l p^l)!}.$$

In particular, we have $q = q_{a,l,1,0}$.

Lemma 2.5. $q_{a,l,1,i+1} \equiv q_{a,l,1,i} \pmod{p}$.

Proof. We have $(0 \leq i < a_1)$

$$\binom{ap + b - ip}{p} = \frac{q_{a,l,1,i}}{q_{a,l,1,i+1}} \binom{(a_1 - i)p}{p}$$

Or equivalently

$$q_{a,l,1,i+1}\binom{ap+b-ip}{p} = q_{a,l,1,i}\binom{(a_1-i)p}{p}.$$

Now

$$\binom{ap+b-ip}{p} = \binom{(a-i)p+b}{p} \equiv a-i \equiv a_1-i \pmod{p},$$

and

$$\binom{(a_1-i)p}{p} \equiv a_1-i \pmod{p}.$$

Therefore

$$q_{a,l,1,i+1}(a_1-i) \equiv q_{a,l,1,i}(a_1-i) \pmod{p}.$$

Since a_1-i with $i=0,1,2,\ldots,a_1-1$ and p are relatively prime, so $q_{a,l,1,i+1} \equiv q_{a,l,1,i} \pmod{p}$. □

Notice that $q_{a,l,1,a_1}$ corresponds to the case where $a_1=0$, $(a_0+a_2p^2+\ldots+a_lp^l)! = q_{a,l,1,a_1}a_0!(a_2p^2)!\ldots(a_lp^l)!$.

Lemma 2.6. *For $n=ap+b=a_{(k)}p^k+b_{(k)}$ with $0\le b_{(k)}\le p^k-1$ and defining ($1\le k\le l$ and $0\le i\le a_k$)*

$$q_{a,l,k,i} = \frac{(a_{(k)}p^k+b_{(k)}-ip^k)!}{a_0!(a_1p)!\ldots((a_k-i)p^k)!\ldots(a_lp^l)!}$$

with $a_k\ge 1$, (where it is understood that when a and l appears together as the two first labels of one $q_{\ldots.}$, it implies that a is given by $a=a_{(1)}=(a_1a_2\ldots a_l)_p=a_1+a_2p+\ldots+a_lp^{l-1}$) we have ($0\le i<a_k$)

$$\binom{a_{(k)}p^k+b_{(k)}-ip^k}{p^k} = \frac{q_{a,l,k,i}}{q_{a,l,k,i+1}}\binom{(a_k-i)p^k}{p^k}$$

Additionally, $q_{a,l,k,i}\equiv q_{a,l,k,i+1} \pmod{p}$.

In particular for $k=0$, we have ($0\le i<a_0$),

$$ap+b = \frac{q_{a,l,0,i}}{q_{a,l,0,i+1}}(a_0-i)$$

with $a_0\ge 1$.

We can prove this lemma by the following a similar reasoning as earlier and hence we omit it here.

Notice that $q=q_{a,l,k,0}$. And q_{a,l,k,a_k} corresponds to the case where $a_k=0$. Also

$$q_{a-i,l,k,0} = q_{a,l,1,i} \equiv q_{a,l,1,i+1} \pmod{p},$$

with $0\le i<a_1$ and $a_1\ge 1$. So, since $q_{a,l,k,j}\equiv q_{a,l,k,j+1} \pmod{p}$ with $0\le j<a_k$, we have $q_{a-i,l,k,j}\equiv q_{a-i,l,k,0}\equiv q_{a,l,1,i}\equiv q_{a,l,1,0} \pmod{p}$ and $q_{a-i,l,k,0}=q_{a-i,l,l,0}\equiv q_{a-i,l,l,j}\equiv q_{a-i,l,l,a_l}\equiv q_{a-i,l-1,k,0} \pmod{p}$.

So $q_{a-i,l,k,0}\equiv q_{a-i,l-1,k,0}\equiv\cdots\equiv q_{a-i,l,k,0}\equiv q_{a-i,1,1,0}\equiv q_{a,1,1,i}\equiv q_{a,1,1,0} \pmod{p}$. Or $q_{a-i,l,k,0}=q_{a,l,1,i}\equiv q_{a,l,1,0}\equiv q_{a,l,k,0} \pmod{p}$.

Finally we have $q=q_{a,l,k,0}\equiv q_{a,1,1,0} \pmod{p}$.

Lemma 2.7. *We have also the congruence $(a_ip^i)!\equiv a_i!p^{a_i(1+p+\ldots+p^{i-1})} \pmod{p}$.*

Proof. We proceed by induction on i.

Indeed, we have $(a_1p)!\equiv 1p\cdot 2p\cdots(a_1-1)p\cdot a_1p \pmod{p}$. So $(a_1p)!\equiv a_1!p^{a_1} \pmod{p}$.

It follows that $(a_2p^2)!=((a_2p)p)!\equiv(a_2p)!p^{a_2p}\equiv a_2!p^{a_2}p^{a_2p} \pmod{p}$. So $(a_2p^2)!\equiv a_2!p^{a_2(1+p)} \pmod{p}$.

Let us assume that $(a_ip^i)!\equiv a_i!p^{a_i(1+p+\ldots+p^{i-1})} \pmod{p}$.

We have $(a_{i+1}p^{i+1})!=((a_{i+1}p^i)p)!\equiv(a_{i+1}p^i)!p^{a_{i+1}p^i}\equiv a_{i+1}!p^{a_{i+1}(1+p+\ldots+p^{i-1})}p^{a_{i+1}p^i} \pmod{p}$. Thus $(a_{i+1}p^{i+1})!\equiv a_{i+1}!p^{a_{i+1}(1+p+\ldots+p^{i-1}+p^i)} \pmod{p}$.

Hence the result follows. □

We now prove Theorem 2.3.

Proof. If $a_i = 0$ for $i \in [[1, l]]$, then $n = a_0$ and we have $n! = qa_0!$ with $q = 1$. So, if $a_i = 0$ for $i \in [[1, l]]$, $q \equiv 1 \pmod{p}$.

Let consider the case where $a_i = 0$ for all $i > 1$, then $n = a_0 + a_1 p$ and we have $n! = qa_0!(a_1 p)!$. Then

$$qa_0! = \frac{n!}{(a_1 p)!} = (a_0 + a_1 p)(a_0 + a_1 p - 1) \dots (a_1 p + 1) = \prod_{r=0}^{a_0 - 1} (a_1 p + a_0 - r).$$

Since $0 < a_0 - r \leq a_0$ for $r \in [[0, a_0 - 1]]$, we obtain

$$qa_0! \equiv \prod_{r=0}^{a_0 - 1} (a_0 - r) \equiv \prod_{r=1}^{a_0} r \equiv a_0! \pmod{p}.$$

Since $a_0!$ and p are relatively prime, we have $q \equiv 1 \pmod{p}$.
Since $q = q_{a,l,k,0} \equiv q_{a,l,1,0} \pmod{p}$, we conclude that $q \equiv 1 \pmod{p}$ whatever n is. □

We come back to the proof of Theorem 1.1.

Proof. Let m, n be two positive integers whose base p expansion with p a prime, are

$$m = a_0 + a_1 p + \ldots + a_k p^k,$$

and

$$n = b_0 + b_1 p + \ldots + b_l p^l,$$

such that $m \geq n$. We assume that $a_i \geq b_i$ with $i = 0, 1, 2, \ldots, \max(k, l)$. We denote $a = \lfloor \frac{m}{p} \rfloor$ and $b = \lfloor \frac{n}{p} \rfloor$. Since $a_i \geq b_i$ with $i = 0, 1, 2, \ldots, \max(k, l)$, we have $a \geq b$. We define

$$a_{\max(k,l)} = \begin{cases} 0 & \text{if} \quad k < l \\ a_k & \text{if} \quad k \geq l \end{cases}$$

and

$$b_{\max(k,l)} = \begin{cases} b_l & \text{if} \quad k \leq l \\ 0 & \text{if} \quad k > l \end{cases}$$

In particular if $\max(k, l) = k$, $b_i = 0$ for $i > l$ and if $\max(k, l) = l$, $a_i = 0$ for $i > k$. Since $m \geq n$, $\max(k, l) = k$. So, we have $a_{\max(k,l)} = a_k$ and $b_{\max(k,l)} = b_k$ with $b_k = 0$ when $l < k$.

Using these

$$m! = q_{a,k,1,0} a_0!(a_1 p)! \dots (a_k p^k)!,$$
$$n! = q_{b,l,1,0} b_0!(b_1 p)! \dots (b_l p^l)!,$$

and

$$(m - n)! = q_{a-b,k,1,0}(a_0 - b_0)!((a_1 - b_1)p)! \dots ((a_k - b_k)p^k)!.$$

We have

$$\binom{m}{n} = \frac{q_{a,k,1,0}}{q_{b,l,1,0} q_{a-b,k,1,0}} \binom{a_0}{b_0} \binom{a_1 p}{b_1 p} \dots \binom{a_{\max(k,l)} p^{\max(k,l)}}{b_{\max(k,l)} p^{\max(k,l)}}.$$

Rearranging

$$q_{b,l,1,0} q_{a-b,k,1,0} \binom{m}{n} = q_{a,k,1,0} \binom{a_0}{b_0} \binom{a_1 p}{b_1 p} \dots \binom{a_{\max(k,l)} p^{\max(k,l)}}{b_{\max(k,l)} p^{\max(k,l)}}.$$

Since $q_{a,k,1,0} \equiv q_{b,l,1,0} \equiv q_{a-b,k,1,0} \equiv 1 \pmod{p}$, we get

$$\binom{m}{n} \equiv \binom{a_0}{b_0} \binom{a_1 p}{b_1 p} \dots \binom{a_{\max(k,l)} p^{\max(k,l)}}{b_{\max(k,l)} p^{\max(k,l)}} \pmod{p}.$$

Notice that if for some i, $a_i = 0$, then $b_i = 0$ since we assume that $a_i \geq b_i$ and $b_i \geq 0$. In such a case $\binom{a_i p^i}{b_i p^i} = \binom{a_i}{b_i} = 1$.

We assume that $a_i \geq 1$. For $k \in [[1, a_i p^i - 1]]$ and for $i \in [[0, \max(k, l)]]$ with $1 \leq a_i \leq p^i - 1$, $\binom{a_i p^i}{k} \equiv 0 \pmod{p}$

Therefore for $a_i = 1$ we have

$$(1+x)^{p^i} = \sum_{k=0}^{p^i} \binom{p^i}{k} x^k \equiv 1 + x^{p^i} \pmod{p},$$

and for any $a_i \in [[1, p^i - 1]]$

$$(1+x)^{a_i p^i} = ((1+x)^{p^i})^{a_i} \equiv (1+x^{p^i})^{a_i} \pmod{p}.$$

Now comparing

$$(1+x)^{a_i p^i} = \sum_{k=0}^{a_i p^i} \binom{a_i p^i}{k} x^k,$$

and

$$(1+x^{p^i})^{a_i} = \sum_{l=0}^{a_i} \binom{a_i}{l} x^{l p^i}$$

we get by taking $k = b_i p^i$ and $l = b_i$,

$$\binom{a_i p^i}{b_i p^i} \equiv \binom{a_i}{b_i} \pmod{p}.$$

Finally we have

$$\binom{m}{n} \equiv \binom{a_0}{b_0}\binom{a_1}{b_1} \dots \binom{a_{\max(k,l)}}{b_{\max(k,l)}} \pmod{p}.$$

□

References

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